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LIMIT STATE OF STRUCTURAL ELEMENTS DURING INELASTIC DEFORMATION

A. F. Nikitenko and B. S. Reznikov

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It is shown that, in the case of an axisymmetric stress state, the solution of the statically definable boundary-value problem for an ideal rigid-plastic body using the Mises–Schleicher strength criterion is extended to the rigid-creep model with any specified creep-rupture strength and corresponds to the limit state of a real creeping body.

Key words: *creep, creep-rupture strength, ideal rigid-plastic and rigid-creep models.*

A procedure of limit equilibrium analysis for structural elements subjected to constant external temperature and force loads was proposed in [1, 2] using Rabotnov kinetic theory which describes all three stages of creep with damage accumulation taken into account from phenomenological positions. According to this procedure, one should first solve the problem under the assumption of steady-state creep and then require that the obtained stationary stress field satisfy the condition of transition of the material of the structural element to the limit state. In the case of a uniformly heated solid (structural element), this condition has the form

$$\sigma_e = \sigma_*, \tag{1}$$

where σ_e is the equivalent stress (a function of the first power which is homogeneous in stress) which is analytically approximated by the corresponding creep-rupture strength criterion and σ_* is the creep-rupture strength of the material. In the following, as the physical relations of the steady-state creep we use the flow law associated with the surface $\sigma_e = \text{const}$:

$$\eta_{ij} = \frac{W}{\sigma_e} \frac{\partial \sigma_e}{\partial \sigma_{ij}}, \qquad W = B\sigma_e^{n+1}, \qquad i = 1, 2, 3, \quad j = 1, 2, 3. \tag{2}$$

Here η_{ij} and σ_{ij} are the components of the creep strain rate of and stress tensors, respectively, B and n are material characteristics, and $W = \sigma_{ij}\eta_{ij}$ is the energy dissipation power in the creeping material. Supplementing Eq. (2) by the equilibrium equations with the corresponding boundary conditions, the Cauchy relations, and the continuity equations for the creep strain rates, we obtain a system of equations for calculating the stress–strain state of an arbitrary body loaded by external forces.

In Eq. (2) we represent the quantity W as $W = B_1(\sigma_e/k)^{n+1}$. Then, $(W/B_1)^{1/(n+1)} = \sigma_e/k$, where $\sigma_e = k$ as $n \to \infty$. Thus, in the case $n \to \infty$, the mathematical model of steady-state creep (2) degenerates into a mathematical model of an ideal rigid-plastic body if $k = \sigma_{\rm y}$ or the model of a rigid-creeping body if $k = \sigma_*$ ($\sigma_{\rm y}$ is the yield point of the material).

For plane problems of inelastic deformation, we formulate the following conclusion. If on the surface of the body, the boundary conditions are specified only in stresses, then supplementing the equilibrium equations by the condition (1) of transition of the material of the body to the limit state, we obtain a statically definable problem which does not depend on the physical relations of inelastic deformation. Thus, it can be argued that the solution of the boundary-value problem for ideal rigid-plastic bodies is extended to the model of rigid-creeping bodies with any specified creep-rupture strength σ_*, and $\sigma_* \geqslant \sigma_{\rm s}$ ($\sigma_{\rm s}$ is the creep strength of the material; therefore,

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; naf@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 51, No. 1, pp. 147–152, January–February, 2010. Original article submitted November 26, 2008; revision submitted March 11, 2009.

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$\sigma_{\mathrm{s}} \leqslant k \leqslant \sigma_{\mathrm{y}}$). This statement was formulated in [3] provided that σ_e is the stress intensity. In [3], it is noted that there is an analogy between the solution of specific problems (beam bending, rod twisting, etc.) under the assumption of steady-state creep of the material with an exponential law and the solution of the same problems under the assumption of ideal rigid-plastic behavior of the material. The analogy lies in the fact that, as the creep exponent tends to infinity, the stress distribution in a uniformly heated body tends to the ideally plastic distribution which is considered limiting in plastic theory and corresponds to a particular value of the external load. It is easy to show that the statement formulated in [3] and based on the analogy between the solutions of the corresponding problems for $n \to \infty$ is a special case of the calculation of structural elements from the limit equilibrium. It should be noted that in [3] there is no formulation of condition (1) or a similar condition of the limit state of a creeping body.

Condition (1) can be used to calculate the limit external load that provides equal strength of the body at any time up to the moment of exhaustion of its carrying capacity [1, 2]. It is obvious that this value depends significantly on the chosen creep-rupture strength criterion. Below we use the Mises–Schleicher criterion, which follows from the generalized criterion [4]

$$\sigma_e = \sigma_i f(\zeta) + \beta\sigma_0. \tag{3}$$

Here $\sigma_0 = \sigma_{ij}\delta_{ij}/3$ is the hydrostatic stress tensor component, $\sigma_i = \sqrt{3s_{ij}s_{ij}/2}$ is the stress intensity, $s_{ij} = \sigma_{ij} - \sigma_0\delta_{ij}$ are the stress deviator components, δ_{ij} is the Kronecker symbol, $\beta \geqslant 0$ is the internal friction coefficient, and ζ is the stress mode angle. In [4], where the function $f(\zeta)$ is approximated by

$$f(\zeta) = [1 + \alpha(\sin 3\zeta)^{\lambda}]^{1/(2\nu)}, \tag{4}$$

a procedure for determining the characteristics α, λ, and ν is described and various cases of transition of criterion (3) to known criteria are considered. In particular, for $\zeta = m\pi/3$ ($m = 0, \pm1, \pm2$), we have $f(m\pi/3) = 1$ and the generalized criterion (3), (4) degenerates into the well-known Mises–Schleicher criterion $\sigma_e = \sigma_i + \beta\sigma_0$.

We write the principal components σ_1, σ_2, and σ_3 of the stress tensor in trigonometric form [5, 6]

$$\begin{gathered} \sigma_1 = (2/3)\sigma_i \sin(\pi/3 - \zeta) + \sigma_0, \qquad \sigma_2 = (2/3)\sigma_i \sin(\zeta) + \sigma_0, \\ \sigma_3 = -(2/3)\sigma_i \sin(\pi/3 + \zeta) + \sigma_0, \end{gathered} \tag{5}$$

and $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ if $-\pi/6 \leqslant \zeta \leqslant \pi/6$, $\sigma_1 \geqslant \sigma_3 \geqslant \sigma_2$ if $-\pi/2 \leqslant \zeta \leqslant -\pi/6$, and so on for the other ranges of ζ. For $\zeta = m\pi/3$ in case of, for example, $-\pi/2 \leqslant \zeta \leqslant -\pi/6$, Eq. (5) leads to

$$\sigma_1 = \sqrt{3}\,\sigma_i/3 + \sigma_0, \qquad \sigma_3 = \sigma_0, \qquad \sigma_2 = -\sqrt{3}\,\sigma_i/3 + \sigma_0, \tag{6}$$

where the maximum shear stress is $\tau_{\max} = \sqrt{3}\,\sigma_i/3$.

It is obvious that the stress state (6) is pure shear with the hydrostatic pressure imposed on it, which is the case for a plane strain state [5, 6].

Using the Mises–Schleicher criterion, condition (1) for the limit state of the body material is written as

$$\sigma_i + \beta\sigma_0 = \sigma_*. \tag{7}$$

In view of (7), relations (2) in the principal axes of the stress tensor in cylindrical coordinates (r, φ, z) become

$$\eta_\varphi = B\sigma_e^n\Big(\frac{3}{2}\,\frac{s_\varphi}{\sigma_i} + \frac{\beta}{3}\Big), \qquad \eta_r = B\sigma_e^n\Big(\frac{3}{2}\,\frac{s_r}{\sigma_i} + \frac{\beta}{3}\Big), \qquad \eta_z = B\sigma_e^n\Big(\frac{3}{2}\,\frac{s_z}{\sigma_i} + \frac{\beta}{3}\Big). \tag{8}$$

It is obvious that $\eta_\varphi + \eta_r + \eta_z = 0$ only for $\beta = 0$. We next perform the limit equilibrium analysis for a plane strain state. For plane strains, $\eta_z = 0$ [5, 6], and, hence,

$$\sigma_z = \frac{\sigma_\varphi + \sigma_r}{2} - \frac{\beta}{3}\,\sigma_i. \tag{9}$$

It is easy to show that $\sigma_0 = \sigma_z + (2/9)\beta\sigma_i$ and the stress intensity is equal to

$$\sigma_i = \frac{3}{2}\,\frac{\beta_0}{\beta}\,(\sigma_\varphi - \sigma_r). \tag{10}$$

In view of (9) and (10), relations (8) finally become

$$\eta_\varphi = \frac{\beta}{2}\Big(\frac{1}{\beta_0} + 1\Big) B\sigma_e^n, \qquad \eta_r = \frac{\beta}{2}\Big(-\frac{1}{\beta_0} + 1\Big) B\sigma_e^n, \qquad \eta_z = 0; \tag{11}$$

$$\sigma_e = \frac{\beta}{\beta_0}\Big(\frac{\sigma_\varphi - \sigma_r}{2} + \beta_0\,\frac{\sigma_\varphi + \sigma_r}{2}\Big), \qquad \beta_0 = \frac{\sqrt{3}\,\beta}{\sqrt{9-\beta^2}}. \tag{12}$$

We note that, in the case $\beta = 0$, relations (8)–(12) reduce to the well-known relations [3, 6], and in this case, $\sigma_e = \sigma_i$.

To determine the stress–strain state of a body and the limit external load, it is necessary to supplement Eqs. (7), (11), and (12) by the equilibrium equation and the continuity equation for the creep strain rate. For structural elements in an axisymmetric stress state with no shear stresses, we have

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0; \tag{13}$$

$$\frac{d\eta_\varphi}{dr} + \frac{\eta_\varphi - \eta_r}{r} = 0. \tag{14}$$

We introduce the stress function $\Phi = \Phi(r)$ such that

$$\sigma_\varphi = \frac{d^2\Phi}{dr^2}, \qquad \sigma_r = \frac{1}{r}\,\frac{d\Phi}{dr}. \tag{15}$$

In this case, the equilibrium equation (13) is identically satisfied. After appropriate standard operations using (11)–(15), we obtain a third-order differential equation (Euler equation) for $\Phi(r)$. Its corresponding characteristic equation has different real roots $k_1 = 0$, $k_2 = \nu_1$, and $k_3 = 2 - \nu_2$, where

$$\nu_1 = \frac{2}{1+\beta_0}, \qquad \nu_2 = \frac{\nu_1}{n}. \tag{16}$$

Consequently, the solution of the Euler equation has the form

$$\Phi(r) = C_1 + C_2 r^{\nu_1} + C_3 r^{2-\nu_2}. \tag{17}$$

In view of (15), from (17) we obtain

$$\begin{aligned} \sigma_\varphi &= \nu_1(\nu_1 - 1)C_2 r^{\nu_1 - 2} + (2-\nu_2)(1-\nu_2)C_3 r^{-\nu_2}, \\ \sigma_r &= \nu_1 C_2 r^{\nu_1-2} + (2-\nu_2)C_3 r^{-\nu_2}. \end{aligned} \tag{18}$$

The constants C_2 and C_3 in (18) are determined from the boundary conditions.

According to the procedure described above, in order to calculate the limit external load acting on a body, it is necessary to require that the stationary stress field (18) satisfy condition (7). Substitution of (18) into (7) yields

$$\frac{(2-\nu_2)(2-\nu_1-\nu_2)}{2-\nu_1}\, C_3 r^{-\nu_2} = \frac{\sigma_*}{\beta}. \tag{19}$$

Equality (19) should be satisfied simultaneously at all points of the body, i.e., it should not depend on r. This is possible if $\nu_2 = 0$, which, in view of (16), is equivalent to the condition $n \to \infty$. Therefore, Eq. (19) leads to

$$2C_3^* = \sigma_*/\beta, \tag{20}$$

where C_3^* is a constant C_3 expressed in terns of the maximum load. Similarly, we denote by C_2^* the constant C_2 expressed in terms of the maximum load. The stresses corresponding to the limit state of the body are found from formulas (18) through the use of (20):

$$\sigma_\varphi = \nu_1(\nu_1 - 1)C_2^* r^{\nu_1-2} + 2C_3^*, \qquad \sigma_r = \nu_1 C_2^* r^{\nu_1 - 2} + 2C_3^*. \tag{21}$$

As noted above, $\sigma_e = k$ as $n \to \infty$; hence, it is obvious that expression (20) for the maximum load acting on a creeping body is also valid for ideal rigid-plastic bodies if the creep-rupture strength of the material is replaced by its yield point. Similarly, the stress relation (21) for the limit state of a creeping body is valid for ideal rigid-plastic bodies if σ_* is replaced by σ_{y}.

If $\beta = 0$, the Mises–Schleicher criterion degenerates into the Mises criterion, and the condition of transition of the body material to the limit state becomes $\sigma_i = \sigma_*$. From (12) and (16), it follows that $\beta_0 = 0$, $\nu_1 = 2$, and $\nu_2 = 2/n$ for $\beta = 0$ ($\nu_2 = 0$ for $n \to \infty$). Hence, the characteristic equation corresponding to the Euler equation for the function $\Phi(r)$ has the following roots: $k_1 = 0$ and $k_2 = k_3 = 2$. The stress function $\Phi(r)$ corresponding to the limit state of the body is written as

$$\Phi(r) = C_1^* + C_2^* r^2 + C_3^* r^2 \ln r,$$

and the stresses are expressed as

$$\sigma_\varphi = 2C_2^* + 3C_3^* + 2C_3^* \ln r, \qquad \sigma_r = 2C_2^* + C_3^* + 2C_3^* \ln r. \tag{22}$$

Taking into account that $\sigma_i = \sqrt{3}\,(\sigma_\varphi - \sigma_r)/2$ and using (22) instead of (20) to calculate the maximum load, from the condition of the limit state $\sigma_i = k$ we obtain the relation

$$2C_3^* = 2k/\sqrt{3}. \tag{23}$$

As an example, we consider the creep of a thick-walled pipe acted upon by internal pressure p. In this case, the boundary conditions are given by

$$\sigma_r(a) = -p, \qquad \sigma_r(b) = 0, \tag{24}$$

where a and b are the inner and outer radii of the pipe, respectively. Substitution of (18) into (24) yields

$$\nu_1 C_2 = -\frac{b^{2-\nu_1} p}{\beta_1^{\nu_2}(\beta_1^{2-\nu_1-\nu_2} - 1)}; \tag{25}$$

$$(2 - \nu_2) C_3 = \frac{b^{\nu_2} p}{\beta_1^{\nu_2}(\beta_1^{2-\nu_1-\nu_2} - 1)}, \tag{26}$$

where $\beta_1 = b/a$.

Denoting the limit pressure p_{**} and using (26) for $\nu_2 = 0$, from (20) we obtain

$$p_{**} = \frac{\beta_1^{2-\nu_1} - 1}{\beta}\,\sigma_*. \tag{27}$$

Using (27), from (25) we calculate $\nu_1 C_2^*$, and thus, from (21) we obtain the stress field for the limit state of the pipe

$$\sigma_\varphi = \frac{1 - (\nu_1 - 1)\rho^{2-\nu_1}}{\beta}\sigma_*, \qquad \sigma_r = -\frac{\rho^{2-\nu_1} - 1}{\beta}\sigma_*, \qquad \rho = \frac{b}{r}. \tag{28}$$

In (27) and (28), replacing σ_* with $\sigma_{\rm y}$, we find the limit pressure and its corresponding stress field in the cross section of the pipe for ideal rigid-plastic deformation of its material.

For $\beta = 0$, using (22) from (24) we obtain

$$2C_3^* = p_{**}/\ln\beta_1, \qquad 2C_2^* = -p_{**}(\ln b + 1/2)/\ln\beta_1,$$

and, thus, from (22) we have

$$\sigma_\varphi = \frac{p_{**}}{\ln\beta_1}\,(1 - \ln\rho), \qquad \sigma_r = -\frac{p_{**}}{\ln\beta_1}\,\ln\rho, \tag{29}$$

and from (23), we have

$$p_{**} = (2/\sqrt{3}\,)k\ln\beta_1. \tag{30}$$

It is obvious that formulas (29) and (30) coincide with the well-known results [3]. We note that relations (29) and (30) follow from (28) and (27) ($\nu_1 \to 2$ as $\beta \to 0$).

Thus, the solution of the boundary-value problem for an ideal rigid-plastic body is completely extended to the rigid-creep model with any specified creep-rupture strength of the material, and vice versa. In this case, the approximations of the equivalent stress σ_e in the condition of the transition to the limit state and the flow law associated with the same surface $\sigma_e = k$ should be the same. The solutions obtained for the rigid-creep model correspond to the limit state of a real creeping body.

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REFERENCES

1. A. F. Nikitenko, "Limit state of a creeping body," in: *Problems of Optimal Design of Structures*, Proc 2nd All-Russian Seminar (Novosibirsk, April 7–9, 1998, Novosibirsk), Univ. of Architecture and Civil Engineering, Novosibirsk (1998), pp. 94–103.
2. A. F. Nikitenko, "Kinetic creep theory and creep-rupture strength analysis for structural elements. 2. Limit state of nonuniformly heated structural elements," *Probl. Prochn.*, No. 6, 5–14 (2005).
3. L. M. Kachanov, *Creep Theory* [in Russian], Fizmatgiz, Moscow (1960).
4. A. F. Nikitenko, A. M. Kovrizhnykh, and I. V. Kucherenko, "Unified (generalized) criterion of material strength. 1," *Izv. Vyssh. Uchebn. Zaved., Striotel'stvovo*, Nos. 11/12, 4–11 (2006).
5. L. M. Kachanov, *Foundations of the Theory of Plasticity*, North-Holland, Amsterdam–London (1971).
6. Yu. N. Rabotnov, *Mechanics of Deformable Bodies* [in Russian], Nauka, Moscow (1988).